

# THREE-PERIODIC HELICES ON ELLIPTIC CURVES AND THEIR ASSOCIATED REGULAR ALGEBRAS

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ABSTRACT. Let  $k$  denote an algebraically closed field of characteristic zero and let  $X$  denote a smooth elliptic curve over  $k$ . Given a three-periodic elliptic helix  $\underline{\mathcal{E}}$  of vector bundles over  $X$  with endomorphism  $\mathbb{Z}$ -algebra  $\text{End } \underline{\mathcal{E}}$  and quadratic cover  $\mathbb{S}^{nc}(\underline{\mathcal{E}})$ , we prove that  $\text{End } \underline{\mathcal{E}}$  is the quotient of  $\mathbb{S}^{nc}(\underline{\mathcal{E}})$  by a degree three family of normal elements, generalizing [10, Theorem 1.2(4)] to the case in which  $\dim \text{End } \underline{\mathcal{E}}_{i,i+1}$  isn't a constant function of  $i$ . We then show that  $\text{End } \underline{\mathcal{E}}$  is noetherian if and only if it has polynomial growth, and in this case, the ranks of any three consecutive bundles in the helix are a Markov triple. Furthermore, in this case  $\mathbb{S}^{nc}(\underline{\mathcal{E}})$  is a noetherian GK-three  $\mathbb{Z}$ -algebra which is Proj-equivalent to an elliptic algebra. We conclude the paper by constructing several new families of elliptic helices with exponential growth.

Throughout this paper, we let  $k$  denote an algebraically closed field of characteristic zero and we let  $X$  denote a smooth elliptic curve over  $k$ .

## 1. INTRODUCTION

In the classic paper [3], Artin-Tate-Van den Bergh studied regular algebras associated to an elliptic curve  $X$  as noncommutative analogues of the polynomial ring in three variables and hence, also noncommutative planes. One way to interpret their construction is that they embed  $X$  “noncommutatively” by replacing the degree three polarizing line bundle with an invertible bimodule  $\mathcal{E}$ . This gives a noncommutative homogeneous coordinate ring, and taking the quadratic cover (that is, retaining only quadratic relations), one obtains a regular algebra. Such algebras have an associated division ring of fractions, which has transcendence degree two, and Artin conjectured in [2], that the only other such division rings are those which are Ore extensions of transcendence degree one fields and those which are finite over their centre.

Inspired by [12], in which the notion of a helix of coherent sheaves over  $\mathbb{P}^2$ , i.e. a planar helix, was defined and studied in order to classify strongly exceptional collections of vector bundles over  $\mathbb{P}^2$  of length three, Bondal and Polishchuk replaced the powers  $\mathcal{E}^{\otimes i}$  of  $\mathcal{E}$  by a sequence of line bundles  $\underline{\mathcal{E}} = (\mathcal{E}_i)_{i \in \mathbb{Z}}$  satisfying a three-periodic helical condition [7] (see Definition 3.3). A new feature that occurs is that the graded homogeneous coordinate ring is now replaced by the endomorphism ring  $\text{End } \underline{\mathcal{E}} = B = \oplus B_{ij}$ ,  $B_{ij} := \text{Hom}_X(\mathcal{E}_{-j}, \mathcal{E}_{-i})$  which is no longer a graded algebra, but rather is a  $\mathbb{Z}$ -algebra.

In [9] and [10], we further generalised Bondal and Polishchuk's work to certain helices of vector bundles over  $X$ . One constraining feature in these papers was that the algebras constructed were all equigenerated in the sense that all the Hilbert

series  $H_{B,n} := \sum \dim_k B_{n,n+i} t^i$  were identical for all  $n$ . In this work, we study the non-equigenerated case in detail.

Key to the analysis is a computation of the Hilbert series, of which there are now three (see Theorem 6.3):

**Theorem 1.1.** *Let  $\underline{\mathcal{E}}$  be an elliptic helix of vector bundles on  $X$ ,  $B = \text{End } \underline{\mathcal{E}}$  and  $\mathbb{S} = \mathbb{S}^{nc}(\underline{\mathcal{E}})$  its quadratic cover. Then the Hilbert series are 3-periodic in the sense that  $H_{B,n+3} = H_{B,n}$ ,  $H_{\mathbb{S},n+3} = H_{\mathbb{S},n}$ . Furthermore,*

$$(1-1) \quad \begin{pmatrix} H_{\mathbb{S},0}(t) \\ H_{\mathbb{S},1}(t) \\ H_{\mathbb{S},2}(t) \end{pmatrix} = \frac{1}{D(t)} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} H_{B,0}(t) \\ H_{B,1}(t) \\ H_{B,2}(t) \end{pmatrix} = \frac{1-t^3}{D(t)} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

where  $D(t)$  is the  $3 \times 3$ -matrix given in (6-5) of Theorem 6.3.

It follows from [7] that  $\mathbb{S}$  is Artin-Schelter regular of global dimension three (see Theorem 3.7), and from our earlier work [10], that  $\text{End } \underline{\mathcal{E}}$  is the quotient of  $\mathbb{S}$  by a degree 3 family of normal elements (see Corollary 6.5). This generalises the fact that elliptic curves embed as a cubic divisor in  $\mathbb{P}^2$  as well as Artin-Tate-Van den Bergh's analogous noncommutative graded result.

Our general setup raises the possibility of generating counter-examples to Artin's conjecture. Towards this end, we examine the noetherian case where one can form rings of fractions. Before describing our main result, we recall that a *Markov triple* is a three-tuple  $(m_1, m_2, m_3)$  of positive integers such that

$$m_1^2 + m_2^2 + m_3^2 = 3m_1m_2m_3.$$

We also recall (for later use) that there is a correspondence between Markov triples and positive integer solutions of

$$(1-2) \quad x^2 + y^2 + z^2 = xyz$$

defined by multiplication by 3 [1, Proposition 2.2].

Our main result is a classification of three periodic elliptic helices  $\underline{\mathcal{E}}$  such that  $\text{End } \underline{\mathcal{E}}$  is noetherian. We accomplish this in Proposition 6.4, Corollary 8.2 and Corollary 9.3 as follows:

**Theorem 1.2.** *Let  $\underline{\mathcal{E}}$  denote a three-periodic elliptic helix. Then  $\text{End } \underline{\mathcal{E}}$  is noetherian if and only if it has polynomial growth. Furthermore, in this case,  $r_0 := \text{rk } \mathcal{E}_0$ ,  $r_1 := \text{rk } \mathcal{E}_1$ ,  $r_2 := \text{rk } \mathcal{E}_2$  form a Markov triple, that is,  $r_0^2 + r_1^2 + r_2^2 = 3r_0r_1r_2$  and*

$$\dim \text{Hom}(\mathcal{E}_1, \mathcal{E}_2) = r_0, \quad \dim \text{Hom}(\mathcal{E}_2, \mathcal{E}_3) = r_1, \quad \dim \text{Hom}(\mathcal{E}_0, \mathcal{E}_1) = r_2.$$

*Lastly,  $\mathbb{S}^{nc}(\underline{\mathcal{E}})$  is a noetherian GK-dimension three  $\mathbb{Z}$ -algebra, which is Proj-equivalent to an elliptic algebra.*

In particular, we do not find a counterexample to Artin's conjecture. The appearance of Markov triples is, in retrospect, not surprising, since mutation of the foundations of a helix yields another helix (see [12]). For essentially the same reason, Markov numbers appear in [19].

We are also able to analyze the planar helices from [12] using our techniques. In particular, we prove, in Section 4, that all planar helices are ample and restrict to elliptic helices. However, the elliptic helices that arise in this manner correspond to the usual embedding of an elliptic curve in  $\mathbb{P}^2$ .

Our final task is to generate further examples of elliptic helices  $\underline{\mathcal{E}}$  of vector bundles with exponential growth. The helix is completely determined, through

a recursive procedure, by the *seed* or *foundation*  $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2)$  which is a triple of stable bundles of increasing slope. However, not all such triples give rise to elliptic helices and determining which ones do involves a priori checking that an infinite number of recursively defined objects in the derived category are actually bundles. In Proposition 10.2, we give a relatively effective sufficient criterion for a seed to generate an elliptic helix. It is in terms of the *numerical seed*  $([\mathcal{E}_0], [\mathcal{E}_1], [\mathcal{E}_2])$  where  $[\mathcal{E}] := \begin{pmatrix} \deg(\mathcal{E}) \\ \mathrm{rk}(\mathcal{E}) \end{pmatrix}$ . It is strong enough to generate the following elliptic helices.

**Proposition 1.3.** *Consider one of the following numerical seeds:*

- (1)  $\left( \begin{pmatrix} -r_0 - r_2 - a \\ r_0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} r_0 + r_2 + a \\ r_2 \end{pmatrix} \right)$  for some  $a, r_0, r_2 \in \mathbb{Z}_{\geq 1}$  such that  $r_0, r_2 \geq a + 1$ ,  $r_0 + r_2 > a^2 + a + 1$  and, of course  $\gcd(r_0 + r_2 + a, r_i) = 1$  for  $i = 0, 2$ ,  
OR
- (2)  $\left( \begin{pmatrix} -d - a - 1 \\ d + a - r \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} d \\ r \end{pmatrix} \right)$  where  $a, d, r \in \mathbb{N}$  are chosen so that  $\gcd(d + a + 1, d + a - r) = 1 = \gcd(d, r)$ . Suppose further that  $d > (a + 1)(2r + 1) + 4$  and  $\min\{d + a - r, r\} > 2$ .

In either case, we obtain an elliptic helix with that numerical seed.

*Notation and conventions:* Throughout,  $\mathbf{C}$  will denote a  $k$ -linear, Hom-finite abelian category. We will often write  ${}^j(\mathcal{A}, \mathcal{B})$  for  $\mathrm{Ext}_{\mathbf{C}}^j(\mathcal{A}, \mathcal{B})$ , and we will let  $\mathrm{Coh}Y$  (resp.  $\mathrm{Qcoh}Y$ ) denote the category of coherent (resp. quasicoherent) sheaves over a variety  $Y$ . If  $\mathbf{D}$  and  $\mathbf{E}$  are categories, we will write  $\mathbf{D} \equiv \mathbf{E}$  if there is an equivalence of categories between  $\mathbf{D}$  and  $\mathbf{E}$ .

## 2. BASIC NOTIONS FROM NONCOMMUTATIVE ALGEBRAIC GEOMETRY

We will show that planar helices and elliptic helices “of Markov type” are ample in Lemma 4.2 and Theorem 8.1, respectively. We will use this fact to prove that the algebras associated to a helix  $\underline{\mathcal{E}}, \mathbb{S}^{nc}(\underline{\mathcal{E}})$  and  $\mathrm{End}(\underline{\mathcal{E}})$ , are noetherian. For the readers convenience, we recall the relevant results from noncommutative algebraic geometry (in the  $\mathbb{Z}$ -algebra context) here.

The definition of ampleness we will need is (superficially) distinct from that in [16], as our indexing conventions are different. We let  $\underline{\mathcal{E}} = (\mathcal{E}_i)_{i \in \mathbb{Z}}$  denote a sequence of objects in  $\mathbf{C}$  such that  $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_i) = k$ .

We call  $\underline{\mathcal{E}}$

- *projective* if for every epimorphism  $f : \mathcal{M} \rightarrow \mathcal{N}$  in  $\mathbf{C}$  there exists an integer  $n$  such that  $\mathrm{Hom}_{\mathbf{C}}(\mathcal{E}_{-i}, f)$  is surjective for all  $i > n$ , and
- *ample* for  $\mathbf{C}$  if it is projective, and if for every  $\mathcal{M} \in \mathbf{C}$  and every  $m \in \mathbb{Z}$  there exists a surjection

$$\bigoplus_{j=1}^s \mathcal{E}_{-i_j} \rightarrow \mathcal{M}$$

for some  $i_1, \dots, i_s$  with  $i_j \geq m$  for all  $j$ .

The significance of this concept for us is that ample sequences for  $\mathbf{C}$  yield homogeneous coordinate rings of  $\mathbf{C}$  [16, Theorem 2.4]. In order to recall this result (in Theorem 2.1), we remind the reader of some terminology from noncommutative algebraic geometry. Suppose  $C$  is a positively graded, connected, locally finite  $\mathbb{Z}$ -algebra. We let  $\mathrm{Gr}C$  denote the category of graded right  $C$ -modules. A graded right  $C$ -module  $M$  is *right bounded* if  $M_n = 0$  for all  $n \gg 0$ . We let  $\mathrm{Tors}C$  denote the

full subcategory of  $\text{Gr}C$  consisting of modules whose elements  $m$  have the property that the right  $C$ -module generated by  $m$  is right bounded. If  $\text{Tors}C$  is a localizing subcategory of  $\text{Gr}C$ , which occurs, for example, if  $C$  is generated in degree one, then we may form the quotient  $\text{Gr}C/\text{Tors}C =: \text{Proj}C$ .

We let  $P_i := \bigoplus_j C_{ij} = e_i C$  where  $e_i$  denotes the unit in the ring  $C_{ii}$ . We say that  $M \in \text{Gr}C$  is *finitely generated* if there is a surjection  $P \rightarrow M$  where  $P$  is a finite direct sum of modules of the form  $P_i$ . We say  $M$  is *coherent* if it is finitely generated and if for every homomorphism  $f : P \rightarrow M$  with  $P$  a finite direct sum of  $P_i$ 's,  $\ker f$  is finitely generated. We denote the full subcategory of  $\text{Gr}C$  consisting of coherent modules by  $\text{coh}C$ . By [16, Proposition 1.1],  $\text{coh}C$  is an abelian subcategory of  $\text{Gr}C$  closed under extensions.

We let  $\text{tors}C$  denote the full subcategory of  $\text{coh}C$  consisting of right-bounded modules. One can check that this is a Serre subcategory of  $\text{coh}C$ . If  $C$  is graded coherent, we define

$$\text{proj}C := \text{coh}C/\text{tors}C,$$

which is abelian.

We will need the following version of [16, Proposition 2.3(ii) and Theorem 2.4] and [5, Theorem 4.5]. In the statement, we use the notation  $C_{\geq 0} := \bigoplus_{i \leq j} C_{ij}$ .

**Theorem 2.1.** *If  $\underline{\mathcal{E}}$  is an ample sequence then  $\text{End}(\underline{\mathcal{E}})_{\geq 0}$  is right coherent, and there is an equivalence*

$$\mathbf{C} \cong \text{proj} \text{End}(\underline{\mathcal{E}})_{\geq 0}.$$

*If furthermore, the category  $\mathbf{C}$  is noetherian, then  $\text{End}(\underline{\mathcal{E}})_{\geq 0}$  is right noetherian.*

*Proof.* The coherence of  $A := \text{End}(\underline{\mathcal{E}})_{\geq 0}$  [16, Proposition 2.3(ii)] whilst the category equivalence is [16, Theorem 2.4]. We now assume that  $\mathbf{C}$  is noetherian and show  $A$  is right noetherian. To this end, let  $\{I_j\}_{j \in \mathbb{N}}$  be an increasing chain of submodules of  $e_i A$ . By the category equivalence and the fact that  $\mathbf{C}$  is noetherian, this chain stabilises in  $\text{proj} A$  so we may as well assume that  $I_{j+1}/I_j$  is torsion for all  $j$  and they all correspond to some subobject  $J < \mathcal{E}_{-i}$ . To the exact sequence

$$0 \rightarrow J \rightarrow \mathcal{E}_{-i} \rightarrow \mathcal{E}_{-i}/J \rightarrow 0,$$

we apply the left exact functor  $\Gamma_{\geq i} = \bigoplus_{n \leq -i} \text{Hom}_{\mathbf{C}}(E_n, -)$  to obtain the exact sequence of  $A$ -modules

$$0 \rightarrow \Gamma_{\geq i} J \rightarrow e_i A \rightarrow \Gamma_{\geq i}(\mathcal{E}_{-i}/J).$$

From the lemma below, we know that  $\Gamma_{\geq i} J$  is saturated in  $e_i A$  so  $\Gamma_{\geq i} J$  contains every  $I_j$  and the quotient is torsion. Also,  $\Gamma_{\geq i} J$  is coherent and in particular, finitely generated by [16, Proposition 2.3] so  $\Gamma_{\geq i} J/I_0$  is finite dimensional as a  $k$ -space and the chain  $\{I_j\}$  must stabilise.  $\square$

**Lemma 2.2.** *If  $\underline{\mathcal{E}}$  is an ample sequence in  $\mathbf{C}$  then  $\Gamma_{\geq i} X$  is torsion-free for every  $X \in \mathbf{C}$ .*

*Proof.* Consider  $0 \neq \phi \in \text{Hom}_{\mathbf{C}}(\mathcal{E}_j, X) \subseteq \Gamma_{\geq i} X$  for some  $j \leq -i$ . By ampleness, given any  $m < j$  we can find  $j_1, \dots, j_l < m$  and a surjection  $\Psi : \bigoplus_r \mathcal{E}_{j_r} \rightarrow \mathcal{E}_j$  so  $\phi \circ \Psi \neq 0$ . It follows that there is a component  $\psi : \mathcal{E}_{j_r} \rightarrow \mathcal{E}_j$  of  $\Psi$  such that  $\phi \circ \psi \neq 0$  so  $\Gamma_{\geq i} X$  is indeed torsion-free.  $\square$

We next examine left finiteness conditions on  $\text{End}(\mathcal{E})_{\geq 0}$ . To this end, we recall (from [15]) that if  $C$  is a  $\mathbb{Z}$ -algebra, then the *opposite  $\mathbb{Z}$ -algebra of  $C$*  is the algebra denoted  $C^o$  and with  $C_{ij}^o := C_{-j, -i}$ , and with multiplication defined as follows: for  $a^o \in C_{jk}^o = C_{-k, -j}$ ,  $b^o \in C_{ij}^o = C_{-j, -i}$ ,  $b^o a^o = ab \in C_{-k, -i} = C_{ik}^o$ . We also note that, by [15, Proposition 2.2], the category of graded right  $C^o$ -modules is equivalent to the category of graded left  $C$ -modules. In the next result, we will invoke the term *duality* on  $\mathbb{C}$ , by which we mean a  $k$ -linear contravariant functor  $D : \mathbb{C} \rightarrow \mathbb{C}$  such that  $D^2 \cong \text{id}_{\mathbb{C}}$ .

**Proposition 2.3.** *Suppose  $D$  is a duality on  $\mathbb{C}$ , and let  $\underline{\mathcal{E}}$  be an ample sequence on  $\mathbb{C}$ . Let  $D(\underline{\mathcal{E}})$  denote the sequence of objects with  $D(\underline{\mathcal{E}})_i := D(\mathcal{E}_{-i})$ . If  $D(\underline{\mathcal{E}})$  is ample, then  $\text{End}(\underline{\mathcal{E}})_{\geq 0}$  is left and right coherent. Furthermore, if  $\mathbb{C}$  is noetherian then  $\text{End}(\underline{\mathcal{E}})_{\geq 0}$  is left and right noetherian.*

*Proof.* We claim that if  $E := \text{End}(\underline{\mathcal{E}})_{\geq 0}$ , then  $E^o \cong \text{End}(D(\underline{\mathcal{E}}))_{\geq 0}$ . From the claim, it will follow that since  $D(\underline{\mathcal{E}})$  is ample,  $E^o$  is right coherent by Theorem 2.1. Thus,  $E$  is left coherent. Similarly, if  $\mathbb{C}$  is noetherian and  $D(\underline{\mathcal{E}})$  is ample, then by the second part of Theorem 2.1,  $E^o$  is right noetherian, so that  $E$  is left noetherian as desired.

It remains to prove the claim. To do so, we note that for each  $i, j$  there is an isomorphism

$$E_{ij}^o = E_{-j, -i} = (\mathcal{E}_i, \mathcal{E}_j) \xrightarrow{D} (D(\mathcal{E}_j), D(\mathcal{E}_i)) = (D(\underline{\mathcal{E}})_{-j}, D(\underline{\mathcal{E}})_{-i}) = \text{End}(D(\underline{\mathcal{E}}))_{ij}.$$

In addition, the above assignment is compatible with multiplication by functoriality and contravariance of  $D$ .  $\square$

We will also need the following  $\mathbb{Z}$ -algebra version of [3, Lemma 8.2]. The statement uses the notion of normal family of elements, which is defined in [10, Definition 8.6].

**Proposition 2.4.** *Let  $C$  be a positively graded, connected locally finite  $\mathbb{Z}$ -algebra and  $x = \{x_i \in e_i C\}_{i \in \mathbb{Z}}$  be a family of positive degree normal elements. If  $C/(x)$  is right noetherian, then so is  $C$ .*

*Proof.* We present here the classic proof of Hilbert's basis theorem adapted to this setting. To simplify notation, we will use  $x^n$  to denote an appropriate legal product of  $n$  of the  $x_i$ 's. The choice of which  $n$ -fold product should be clear from the context. For example, if  $c \in C_{ij}$ , then the left-most  $x$  in  $cx^n$  lies in  $e_j C$ .

We show that any submodule  $I < e_i C$  is finitely generated. Consider the “ $x$ -saturation”

$$\bar{I} = \{c \in e_i C \mid cx^N \in I, N \gg 0\}.$$

Normality of  $x$  ensures that  $\bar{I}$  is a submodule of  $e_i C$ . Now  $e_i C / e_i x C$  is noetherian so we can find generators  $a_1, \dots, a_n$  for  $\bar{I}$  modulo  $e_i x C$ . Pick  $l_1, \dots, l_n \in \mathbb{N}$  so that  $a_1 x^{l_1}, \dots, a_n x^{l_n} \in I$  and let  $l = \max\{l_i\}$ . Let  $b_1, \dots, b_m \in I$  be generators for  $I$  modulo  $e_i x^l C$ .

We complete the proof by showing that  $a'_1 := a_1 x^{l_1}, \dots, a'_n := a_n x^{l_n}, b_1, \dots, b_m$  generate  $I$ . Let  $c \in I \cap e_i x^n C$ . It suffices by induction to find  $c' \in I \cap e_i x^{n+1} C$  such that  $c - c'$  is a  $C$ -linear combination of the  $a'_i, b_j$  since the bidegree of  $c$  and  $c'$  are fixed. If  $n < l$ , then it suffices to use a linear combination of the  $b_i$ 's. If  $n \geq l$ , we write  $c = c'' x^n$  where  $c'' \in \bar{I}$  so we can write  $c'' = \sum a_i c_i + c''' x$ . Using normality of  $x$ , we are done.  $\square$

## 3. HELICES

The main focus of this paper is the study of helices. In this section, we collect the relevant definitions of the term “helix” we will need. In particular, we define weak helices, and recall the notions of elliptic helices and planar helices, from [7, Section 7] and [12], respectively. In the next section, we relate the latter two notions.

**3.1. Weak helices.** We first define the notion of “weak helix”, which is a generalization of the other notions of helix we will study. To this end, let  $V$  be a finite dimensional  $k$ -space and  $\mathcal{M} \in \mathbf{C}$ .

*Definition 3.1.* Since  $\mathbf{C}$  has finite direct sums, there exists an object  $V \otimes_k \mathcal{M} \in \mathbf{C}$  such that

$$\mathrm{Hom}_{\mathbf{C}}(V \otimes_k \mathcal{M}, -) = V^* \otimes_k \mathrm{Hom}_{\mathbf{C}}(\mathcal{M}, -).$$

It is unique up to canonical isomorphism.

This definition has some advantages over a similar definition given in [4] such as the following. Suppose  $\mathcal{L} \in \mathbf{C}$  and let  $V = \mathrm{Hom}_{\mathbf{C}}(\mathcal{M}, \mathcal{L})$ . There is a canonical element

$$1 \in \mathrm{End}_k V \stackrel{\mathrm{can}}{\cong} V^* \otimes_k \mathrm{Hom}_{\mathbf{C}}(\mathcal{M}, \mathcal{L})$$

which by Definition 3.1 gives the *evaluation map*

$$\mathrm{Hom}(\mathcal{M}, \mathcal{L}) \otimes_k \mathcal{M} \rightarrow \mathcal{L}.$$

*Definition 3.2.* An ordered pair  $(\mathcal{E}, \mathcal{F})$  of objects in  $\mathbf{C}$  is said to be *left mutable* if  $j(\mathcal{E}, \mathcal{F}) = 0$  for  $j \neq 0$  and furthermore the evaluation map

$$\epsilon: \mathrm{Hom}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \rightarrow \mathcal{F}$$

is surjective. In this case we define the *left mutation*  $L_{\mathcal{E}}\mathcal{F} := \ker \epsilon$ . We also say  $\mathcal{F}$  *left mutates through  $\mathcal{E}$  in  $\mathbf{C}$* .

The right-handed versions are defined similarly. Explicitly, the pair  $(\mathcal{E}, \mathcal{F})$  is *right mutable* if  $j(\mathcal{E}, \mathcal{F}) = 0$  for  $j \neq 0$  and furthermore the coevaluation map

$$\eta: \mathcal{E} \rightarrow {}^* \mathrm{Hom}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{F}$$

is injective. In this case we define the *right mutation*  $R_{\mathcal{F}}\mathcal{E} := \mathrm{cok} \eta$ .

For the rest of this section, we assume we have a sequence  $\underline{\mathcal{E}} = (\mathcal{E}_i)_{i \in \mathbb{Z}}$  of objects in  $\mathbf{C}$  such that  $\Delta_j := \dim \mathrm{Hom}(\mathcal{E}_{j-1}, \mathcal{E}_j) \neq 0$ . Suppose, for all  $i \in \mathbb{Z}$ :

- (1) the pair  $(\mathcal{E}_i, \mathcal{E}_{i+1})$  is right mutable,
- (2) the pair  $(R_{\mathcal{E}_{i+1}}\mathcal{E}_i, \mathcal{E}_{i+2})$  is right mutable and  $R_{\mathcal{E}_{i+2}}R_{\mathcal{E}_{i+1}}\mathcal{E}_i \cong \mathcal{E}_{i+3}$ ,
- (3)  ${}^* \mathrm{Hom}(R_{\mathcal{E}_{i+1}}\mathcal{E}_i, \mathcal{E}_{i+2}) \cong \mathrm{Hom}(\mathcal{E}_{i+2}, \mathcal{E}_{i+3})$ .

*Definition 3.3.* Sequences  $\underline{\mathcal{E}}$  satisfying (1) and (2) above will be called *weak helices of period three, three-periodic weak helices*, or just *weak helices*. We will also say that  $\underline{\mathcal{E}}$  is *three-helical* in this case.

By property (1), there is an exact sequence

$$0 \longrightarrow \mathcal{E}_i \longrightarrow {}^* \mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_{i+1}) \otimes \mathcal{E}_{i+1} \longrightarrow R_{\mathcal{E}_{i+1}}\mathcal{E}_i \longrightarrow 0,$$

and by (2), there is an exact sequence

$$0 \longrightarrow R_{\mathcal{E}_{i+1}}\mathcal{E}_i \longrightarrow {}^* \mathrm{Hom}(R_{\mathcal{E}_{i+1}}\mathcal{E}_i, \mathcal{E}_{i+2}) \otimes \mathcal{E}_{i+2} \longrightarrow \mathcal{E}_{i+3} \longrightarrow 0.$$

Splicing these sequences together and utilizing (3) yields an exact sequence (3-1)

$$0 \longrightarrow \mathcal{E}_i \longrightarrow {}^* \text{Hom}(\mathcal{E}_i, \mathcal{E}_{i+1}) \otimes \mathcal{E}_{i+1} \longrightarrow \text{Hom}(\mathcal{E}_{i+2}, \mathcal{E}_{i+3}) \otimes \mathcal{E}_{i+2} \longrightarrow \mathcal{E}_{i+3} \longrightarrow 0.$$

Note that condition (3) above essentially means that  $R_{\mathcal{E}_{i+1}} \mathcal{E}_i \simeq L_{\mathcal{E}_{i+2}} \mathcal{E}_{i+3}$  so there is a symmetry between left and right mutations.

**3.2. Elliptic helices.** We now define elliptic helices. We will study this concept in more depth in Sections 5, 6, 7 and 8.

*Definition 3.4.* An object  $\mathcal{E} \in \mathcal{C}$  is *elliptically exceptional* if i)  ${}^j(\mathcal{E}, \mathcal{E}) \cong k$  for  $j = 0, 1$  and is zero otherwise and ii) for any  $\mathcal{F} \in \mathcal{C}$ , the natural pairing

$${}^0(\mathcal{E}, \mathcal{F}) \otimes {}^1(\mathcal{F}, \mathcal{E}) \rightarrow {}^1(\mathcal{E}, \mathcal{E}) = k$$

is non-degenerate.

It follows (see [10, Section 6]) that a coherent sheaf  $\mathcal{E}$  over  $X$  is elliptically exceptional in  $\text{Coh} X$  if and only if  $\mathcal{E}$  is either a vector bundle with relatively prime degree and rank or  $\mathcal{E}$  is a skyscraper sheaf.

Our abelian category version of Bondal-Polishchuk's three-periodic elliptic helices is the same one used in [10]:

*Definition 3.5.* ([7, Section 7, p. 250]) Let  $\underline{\mathcal{E}} = (\mathcal{E}_i)_{i \in \mathbb{Z}}$  be a sequence of elliptically exceptional objects in  $\mathcal{C}$ . We say  $\underline{\mathcal{E}}$  is a *three-periodic elliptic helix* or just an *elliptic helix* if  $\underline{\mathcal{E}}$  is a three-periodic weak helix such that for all  $i < j$  we have  ${}^l(\mathcal{E}_i, \mathcal{E}_j) = 0$  for  $l \neq 0$ .

- Remark 3.6.*
- (1) By the remark following [10, Question 7.1], if  $\underline{\mathcal{E}}$  is an elliptic helix, then  $\mathcal{E}_i$  is a simple and hence stable bundle for all  $i \in \mathbb{Z}$ . Furthermore, the slopes of these bundles is increasing.
  - (2) The analogous definition in terms of left mutations recovers the same concept.
  - (3) Property (3) preceding the definition of weak helix automatically holds for an elliptic helix [13, Lemma 3].
  - (4) Any weak helix of stable bundles on an elliptic curve  $X$  is automatically an elliptic helix since the stable bundles must have strictly increasing slope so  ${}^1(\mathcal{E}_i, \mathcal{E}_j) = 0$  for  $i < j$ .

The significance of this definition is the following result from [7], adapted to our context ([10, Theorem 5.4]):

**Theorem 3.7** ([7]). *Let  $\underline{\mathcal{E}}$  be an three-periodic elliptic helix. Let  $\mathbb{S}^{nc}(\underline{\mathcal{E}})$  be the quadratic part of the endomorphism algebra  $\text{End}(\underline{\mathcal{E}})$ . Then the following hold:*

- (1) *The algebra  $\mathbb{S}^{nc}(\underline{\mathcal{E}})$  is 3-periodic, Koszul, has global dimension three, and is AS-regular of dimension three and Gorenstein parameter three (in the sense of [14, Definition 7.1]).*
- (2) *The canonical map  $\mathbb{S}^{nc}(\underline{\mathcal{E}}) \longrightarrow \text{End}(\underline{\mathcal{E}})$  is surjective so in particular,  $\text{End}(\underline{\mathcal{E}})$  is generated in degree one.*

**3.3. Planar helices.** We end this section by giving the definition of planar helix from [6, Definition p.80], which we will study in detail in Section 4. First, we need to recall that a coherent sheaf  $\mathcal{E}$  on a variety over  $k$  is called *exceptional* if  $\dim^0(\mathcal{E}, \mathcal{E}) = 1$  and  $\dim^i(\mathcal{E}, \mathcal{E}) = 0$  for  $i > 0$ .

*Definition 3.8.* A sequence  $\underline{\mathcal{E}}$  of exceptional bundles on  $\mathbb{P}^2$  is a *planar helix* if it is a three-periodic weak helix which further satisfies i) the periodicity relation  $\mathcal{E}_{i-3} \simeq \mathcal{E}_i \otimes \omega_{\mathbb{P}^2}$  and ii) any three consecutive terms  $\mathcal{E}_i, \mathcal{E}_{i+1}, \mathcal{E}_{i+2}$  is exceptional in the sense that  $\mathbf{R}\mathrm{Hom}(\mathcal{E}_j, \mathcal{E}_i) = 0$  whenever  $i \leq l < j < i + 2$ .

#### 4. PLANAR HELICES AND THEIR ASSOCIATED $\mathbb{Z}$ -ALGEBRAS

In this section, we show that planar helices are ample, have noetherian endomorphism rings, and restrict to elliptic helices as defined in Section 3.

**Lemma 4.1.** *Let  $\underline{\mathcal{V}} = (\mathcal{V}_i)$  be a sequence of non-zero vector bundles on a projective scheme  $X$  such that for any  $\mathcal{M} \in \mathrm{Coh}(X)$ , we have  $\mathrm{Ext}_X^1(\mathcal{V}_i, \mathcal{M}) = 0$  for  $i \ll 0$ . Then  $\underline{\mathcal{V}}$  is ample in  $\mathrm{Coh}(X)$ .*

*Proof.* Projectivity follows from our hypotheses and the long exact sequence in  $\mathrm{Ext}$ . We show now there exists a surjection of the form  $\bigoplus_j \mathcal{V}_{i_j} \rightarrow \mathcal{M}$  where the  $i_j$  can be made as negative as you like. We pick a closed point  $x \in X$  and let  $\mathcal{K} := \ker(\mathcal{M} \rightarrow \mathcal{M} \otimes k(x))$ . Pick  $i \ll 0$  such that  $\mathrm{Ext}_X^1(\mathcal{V}_i, \mathcal{K}) = 0$  so by projectivity, we may lift any surjection  $\bar{\phi}: \mathcal{V}_i^r \rightarrow \mathcal{M} \otimes k(x)$  to a morphism  $\phi: \mathcal{V}_i^r \rightarrow \mathcal{M}$ . Now  $\phi$  is surjective in a neighbourhood of  $x$  by Nakayama's lemma so we are done by dévissage.  $\square$

**Lemma 4.2.** *Let  $\mathcal{V}$  be a vector bundle on projective scheme  $X$  and  $\mathcal{L}$  an ample line bundle. Then the pair  $(\mathcal{V}, (-) \otimes_X \mathcal{L})$  is ample in the sense of [5, (4.2.1)], that is,  $(\mathcal{V} \otimes_X \mathcal{L}^{\otimes i})$  is ample.*

*Proof.* By Lemma 4.1, we need only prove projectivity and so consider an exact sequence of coherent sheaves

$$0 \rightarrow \mathcal{M}' \rightarrow \mathcal{M} \rightarrow \mathcal{M}'' \rightarrow 0.$$

Projectivity follows from

$$\mathrm{Ext}_X^1(\mathcal{V} \otimes \mathcal{L}^{\otimes -i}, \mathcal{M}') = H^1(X, \mathcal{V}^* \otimes \mathcal{M}' \otimes \mathcal{L}^{\otimes i}) = 0$$

for  $i \gg 0$ .  $\square$

**Corollary 4.3.** *Let  $\underline{\mathcal{E}}$  be a planar helix. Then*

- (1) *The  $\mathbb{Z}$ -algebra  $\mathrm{End}(\underline{\mathcal{E}})$  is noetherian,*
- (2) *The canonical map  $\mathbb{S}^{nc}(\underline{\mathcal{E}}) \rightarrow \mathrm{End}(\underline{\mathcal{E}})$  is an isomorphism,*
- (3) *There are equivalences of categories*

$$\mathrm{proj}\mathbb{S}^{nc}(\underline{\mathcal{E}}) \cong \mathrm{Coh}\mathbb{P}^2$$

and

$$\mathrm{Proj}\mathbb{S}^{nc}(\underline{\mathcal{E}}) \cong \mathrm{Qcoh}\mathbb{P}^2.$$

*Proof.* To prove (1), we use Lemma 4.2 to see  $\underline{\mathcal{E}}$  and its dual are ample so (1) now follows from Proposition 2.3.

Part (2) follows from the fact that  $\mathrm{End}(\underline{\mathcal{E}})$  is quadratic by [7, Theorem 4.2]. The first part of (3) now follows from (2) and the fact that  $\underline{\mathcal{E}}$  is ample for  $\mathrm{Coh}\mathbb{P}^2$ . To prove the second part of (3), we note that by (1) and [17, Theorem 8.9], the category  $\mathrm{Proj}\mathbb{S}^{nc}(\underline{\mathcal{E}})$  can be recovered up to equivalence by its full subcategory of noetherian objects,  $\mathrm{proj}\mathbb{S}^{nc}(\underline{\mathcal{E}})$ , and similarly  $\mathrm{Qcoh}\mathbb{P}^2$  can be recovered up to equivalence by  $\mathrm{Coh}\mathbb{P}^2$ .  $\square$

We now show that planar helices restrict to elliptic helices.

**Proposition 4.4.** *Let  $\underline{\mathcal{E}}$  be a planar helix and  $X \subset \mathbb{P}^2$  be a smooth cubic. Then  $\underline{\mathcal{E}}|_X := (\mathcal{E}_i|_X)$  is an elliptic helix on  $X$ .*

*Proof.* We first show that the restriction map induces isomorphisms  $(\mathcal{E}_{i-1}, \mathcal{E}_i) \simeq (\mathcal{E}_{i-1}|_X, \mathcal{E}_i|_X)$ . Consider the exact sequence

$$(4-1) \quad 0 \rightarrow \mathcal{E}_i(-3) \rightarrow \mathcal{E}_i \rightarrow \mathcal{E}_i|_X \rightarrow 0.$$

We deduce the exact sequence

$$0 \rightarrow (\mathcal{E}_{i-1}, \mathcal{E}_i(-3)) \rightarrow (\mathcal{E}_{i-1}, \mathcal{E}_i) \rightarrow (\mathcal{E}_{i-1}, \mathcal{E}_i|_X) \rightarrow {}^1(\mathcal{E}_{i-1}, \mathcal{E}_i(-3)).$$

The desired isomorphism follows from the fact that  $\mathcal{E}_i(-3) \simeq \mathcal{E}_{i-3}$  and the fact that  $\{\mathcal{E}_{i-3}, \mathcal{E}_{i-2}, \mathcal{E}_{i-1}\}$  is exceptional.

Now  $R_{\mathcal{E}_i}\mathcal{E}_{i-1}$  is a bundle so the right mutation sequence

$$0 \rightarrow \mathcal{E}_{i-1} \rightarrow {}^*(\mathcal{E}_{i-1}, \mathcal{E}_i) \otimes \mathcal{E}_i \rightarrow R_{\mathcal{E}_i}\mathcal{E}_{i-1} \rightarrow 0$$

restricts to a right mutation sequence on  $X$  and similarly for left mutations. It now follows that  $\underline{\mathcal{E}}|_X$  is also a weak helix.

Applying  $\mathbf{R}\mathrm{Hom}(\mathcal{E}_i, -)$  to (4-1) similarly shows that  $\mathrm{End} \mathcal{E}_i|_X = k$ . Remark 3.6(4) completes the proof.  $\square$

## 5. ON THE NUMERICS OF THREE-PERIODIC ELLIPTIC HELICES

We now investigate the numerical properties of three-periodic elliptic helices  $\underline{\mathcal{E}}$  on an elliptic curve  $X$ . A key invariant is a triple  $\vec{\Delta}$  recording the dimensions of  $\mathrm{End} \underline{\mathcal{E}}$  in degree one. In this section, we analyse the numerics of helices to show only ‘‘Markov’’ triples  $\vec{\Delta}$  can give rise to indexed algebras of polynomial growth. We enlarge our scope in this section to include objects in  $D^b(X)$ , as many of the computations work in this context.

We have an isomorphism  $(\mathrm{deg}, \mathrm{rk}): K_0(X) \rightarrow \mathbb{Z}^2$ . For objects  $E, E' \in D^b(X)$  we let  $[E], [E']$  denote their images in  $K_0(X)$  and

$$(5-1) \quad \chi([E], [E']) = \chi(E, E') := \chi(\mathbf{R}\mathrm{Hom}(E, E')) = \begin{vmatrix} \mathrm{deg} E' & \mathrm{deg} E \\ \mathrm{rk} E' & \mathrm{rk} E \end{vmatrix}.$$

Let  $E_0, E_1, E_2 \in D^b(X)$  and consider the right mutation  $E'_2 := R_{E_1}E_0$  (defined in terms of exact triangles, see [7, Proposition 7.1]) which has

$$[E'_2] = \chi(E_0, E_1)[E_1] - [E_0].$$

If we write

$$\Delta_{ij} = \chi(E_i, E_j)$$

then we have the iterated right mutation  $E_3 := R_{E_2}E'_2$  satisfies

$$[E_3] = \chi(E'_2, E_2)[E_2] - [E'_2] = (\Delta_{01}\Delta_{12} - \Delta_{02})[E_2] - \Delta_{01}[E_1] + [E_0].$$

Applying  $\chi(E_2, -)$  to both sides we obtain the formulas

$$(5-2) \quad \Delta_{23} = \Delta_{01}\Delta_{12} - \Delta_{02}, \quad [E_3] = [E_0] - \Delta_{01}[E_1] + \Delta_{23}[E_2].$$

We may repeat the process forwards by mutating  $E_1$  past  $E_2$  and  $E_3$  to define  $E_4$ , and backwards by mutating  $E_2$  left past  $E_1$  and  $E_0$  to get  $E_{-1}$  and so on. If all these objects turn out to be vector bundles, then we obtain an elliptic helix of period three in  $\mathrm{Coh}X$  as defined in Definition 3.5.

*Definition 5.1.* The sequences  $\underline{E} = \{E_i\}_{i \in \mathbb{Z}}$  generated by a seed  $(E_0, E_1, E_2)$  by mutation as above is said to satisfy the *three-helical condition*. We call  $\underline{E}$  the *three-helix in  $D^b(X)$  generated by the seed*.

We abbreviate  $\Delta_{i-1,i} =: \Delta_i$  and introduce

$$A_j := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -\Delta_j & \Delta_{j+2} \end{pmatrix}$$

so (5-2) gives

$$(5-3) \quad A_i \begin{pmatrix} [E_{i-1}] \\ [E_i] \\ [E_{i+1}] \end{pmatrix} = \begin{pmatrix} [E_i] \\ [E_{i+1}] \\ [E_{i+2}] \end{pmatrix}$$

We let  $A := A_3 A_2 A_1$ .

The following is [10, Corollary 7.11] which is easily reproved with our current notation.

**Proposition 5.2.** *We have  $\Delta_{i+3} = \Delta_i$  for all  $i$  and hence  $A_{i+3} = A_i$ .*

*Proof.* Applying  $\chi(E_1, -)$  to (5-2) now gives

$$-\Delta_1 + \Delta_2 \Delta_3 = \Delta_{13} = \Delta_2 \Delta_3 - \Delta_4$$

so  $\Delta_4 = \Delta_1$ . □

The following is elementary using the 3-periodicity of the  $\Delta_i$ 's.

**Proposition 5.3.**  *$\det(A_i) = 1$  and  $A_i$  cyclically permutes the coordinates of  $\begin{pmatrix} \Delta_{i+1} \\ \Delta_{i+2} \\ \Delta_i \end{pmatrix}$  for any  $i$ . In particular  $A := A_3 A_2 A_1$  has determinant 1 and eigenvector  $\begin{pmatrix} \Delta_2 \\ \Delta_3 \\ \Delta_1 \end{pmatrix}$  with eigenvalue 1.*

As one can check by explicit calculation, we have

$$(5-4) \quad A = \begin{pmatrix} 1 & -\Delta_1 & \Delta_3 \\ \Delta_1 & 1 - \Delta_1^2 & \Delta_1 \Delta_3 - \Delta_2 \\ \Delta_1 \Delta_2 - \Delta_3 & \Delta_1 \Delta_3 + \Delta_2(1 - \Delta_1^2) & \Delta_2(\Delta_1 \Delta_3 - \Delta_2) + 1 - \Delta_3^2 \end{pmatrix}.$$

This shows the other eigenvalues of  $A$  are

$$(5-5) \quad \lambda_{\pm} := \frac{1}{2} \left( 2 + \Delta_1 \Delta_2 \Delta_3 - (\Delta_1^2 + \Delta_2^2 + \Delta_3^2) \pm \sqrt{(2 + \Delta_1 \Delta_2 \Delta_3 - (\Delta_1^2 + \Delta_2^2 + \Delta_3^2))^2 - 4} \right).$$

**Proposition 5.4.** *The eigenvalues of  $A$  have modulus 1 iff*

$$\Delta_1^2 + \Delta_2^2 + \Delta_3^2 - \Delta_1 \Delta_2 \Delta_3 = 0, 1, 2, 3 \text{ or } 4.$$

*In this case the eigenvalues are 1 and (a) 1, 1 (b)  $e^{\pm i\pi/3}$  (c)  $\pm i$  (d)  $e^{\pm i2\pi/3}$  (e)  $-1, -1$  respectively.*

*Proof.* The sum of the remaining two eigenvalues of  $A$  has to have magnitude  $\leq 2$  since  $\lambda_+ + \lambda_- = 2 \cos \theta$  for some  $\theta \in \mathbb{R}$ , which gives the 5 possibilities above. □

Calculating directly with the explicit form of  $A$  given in (5-4) also gives the following.

**Lemma 5.5.** *Suppose  $\vec{\nabla} := \begin{pmatrix} \Delta_2 \\ \Delta_3 - \Delta_1 \Delta_2 \\ \Delta_1 \end{pmatrix}$ . Then*

- (1)  $\vec{\nabla}^T(A - I) = 0$ , and
- (2) if  $\Delta_1 \neq 0$ , then  $\text{rk}(A - I) = 2$ .

We will be interested in the case where  $\underline{\mathcal{E}} = (\mathcal{E}_i)_{i \in \mathbb{Z}}$  is an elliptic helix. In this case  $\text{End}(\underline{\mathcal{E}})$  is a positively  $\mathbb{Z}$ -indexed algebra and for  $i < j$  we have  $\dim \text{Hom}(\mathcal{E}_i, \mathcal{E}_j) = \chi(\mathcal{E}_i, \mathcal{E}_j)$ . We are interested in the Hilbert series of  $\text{End}(\underline{\mathcal{E}})$  and hence introduce the following

*Definition 5.6.* An elliptic helix  $\underline{\mathcal{E}}$  is said to have *polynomial growth* if there is a polynomial  $p(n)$  such that for all  $i < j$ , we have  $\chi(\mathcal{E}_i, \mathcal{E}_j) \leq p(j - i)$ . If on the other hand, for any  $i$  we have  $\overline{\lim}_{n \rightarrow \infty} \chi(\mathcal{E}_i, \mathcal{E}_{i+n})^{1/n} > 1$ , then we will say that  $\underline{\mathcal{E}}$  has *exponential growth*.

Consider now an elliptic helix  $\underline{\mathcal{E}}$ . We will study the ‘‘Markov’’ case where  $\Delta_1^2 + \Delta_2^2 + \Delta_3^2 = \Delta_1 \Delta_2 \Delta_3$ , that is, all eigenvalues of  $A$  are 1, in detail later. Suppose now that this is not the case and furthermore that we are not in case (e) of Proposition 5.4. Then, the fact that  $\lambda_+ \lambda_- = 1$  ensures the eigenvalues  $1, \lambda_+, \lambda_-$  are all distinct. Fix  $i = 0, 1$  or  $2$ . Then eigenvalue theory gives constants  $\delta_1, \delta_+, \delta_-, \rho_1, \rho_+, \rho_-$  such that

$$(5-6) \quad \begin{pmatrix} \deg(\mathcal{E}_{i+3n}) \\ \text{rk}(\mathcal{E}_{i+3n}) \end{pmatrix} = \begin{pmatrix} \delta_1 + \lambda_+^n \delta_+ + \lambda_-^n \delta_- \\ \rho_1 + \lambda_+^n \rho_+ + \lambda_-^n \rho_- \end{pmatrix}$$

**Proposition 5.7.** *Let  $\underline{\mathcal{E}}$  be an elliptic helix. Then either  $\underline{\mathcal{E}}$  has exponential growth or we must be in the Markov case where  $\Delta_1^2 + \Delta_2^2 + \Delta_3^2 = \Delta_1 \Delta_2 \Delta_3$  in which case  $\underline{\mathcal{E}}$  has polynomial growth.*

*Proof.* Suppose first that  $A$  is diagonalisable so we can use Equation (5-6) above. Let  $j \in \{0, 1, 2\}$  and  $d_j = \deg(\mathcal{E}_j), r_j = \text{rk}(\mathcal{E}_j)$ . Then

$$(5-7) \quad \chi(\mathcal{E}_j, \mathcal{E}_{j+3n}) = \begin{vmatrix} \delta_1 & d_j \\ \rho_1 & r_j \end{vmatrix} + \lambda_+^n \begin{vmatrix} \delta_+ & d_j \\ \rho_+ & r_j \end{vmatrix} + \lambda_-^n \begin{vmatrix} \delta_- & d_j \\ \rho_- & r_j \end{vmatrix}.$$

Now  $\chi(\mathcal{E}_j, \mathcal{E}_{j+3n}) > 0$  for  $n > 0$  and  $\chi(\mathcal{E}_j, \mathcal{E}_{j+3n}) < 0$  for  $n < 0$ . This forces the second and third determinants above to be non-zero. If  $\lambda_{\pm}$  are real and not  $\pm 1$ , then we have exponential growth since we may assume  $\lambda_+ > 1$ . Otherwise, by Proposition 5.4,  $\lambda_{\pm}$  are roots of unity so  $\chi(\mathcal{E}_j, \mathcal{E}_{j+3n})$  is periodic in  $n$  and we cannot have  $\chi(\mathcal{E}_j, \mathcal{E}_{j+3n}) > 0$  for  $n > 0$  and  $\chi(\mathcal{E}_j, \mathcal{E}_{j+3n}) < 0$  for  $n < 0$ .

Now suppose that  $A$  is not diagonalisable. Assuming we are not in the Markov case, then  $-1$  must be an eigenvalue corresponding to a  $2 \times 2$  Jordan block for  $A$ . Repeating the above analysis we find this time that there are three determinants  $D_1, D_{-1}, D'_{-1}$  such that

$$\chi(\mathcal{E}_j, \mathcal{E}_{j+3n}) = D_1 + (-1)^n D_{-1} + (-1)^{n-1} n D'_{-1}.$$

Once again, it is not possible that  $\chi(\mathcal{E}_j, \mathcal{E}_{j+3n})$  always has the same sign as  $n$ .

Finally, suppose we are in the Markov case so  $A$  only has 1 as its sole eigenvalue. Using the appropriate Jordan canonical form for  $A$ , we find scalars  $D, D', D''$  such that  $\chi(\mathcal{E}_i, \mathcal{E}_{i+3n}) = D + nD' + n^2D''$  so we are done. (In fact  $D''$  must also be zero so the growth is linear).  $\square$

## 6. HILBERT SERIES OF REGULAR ALGEBRAS ASSOCIATED TO ELLIPTIC HELICES

In this section, we continue our study of an elliptic helix  $\underline{\mathcal{E}}$  on an elliptic curve  $X$  and use the notation in Section 5. We will compute the Hilbert series of the algebras  $\text{End } \underline{\mathcal{E}}$  and its quadratic cover  $\mathbb{S} = \mathbb{S}^{nc}(\underline{\mathcal{E}})$ . The method here differs from the direct ad hoc calculation for the “equigenerated” case of elliptic helices in [10, Section 8]. The method also gives a nice explanation of the relationship between the two Hilbert series since they are both computed from the corresponding helices.

We work with relative Hilbert series as defined in [10, Definition 8.4], the relevant ones of which are the following.

$$(6-1) \quad H_{\mathbb{S},n}(t) := \sum_{i \in \mathbb{Z}} \dim_k(e_n \mathbb{S}^{nc}(\underline{\mathcal{E}}) e_{n+i}) t^i$$

$$(6-2) \quad H_{\mathcal{E},n}(t) := \sum_{i \in \mathbb{Z}} \dim \text{Hom}_X(\mathcal{E}_{-n-i}, \mathcal{E}_{-n}) t^i.$$

The latter is the Hilbert series for  $\text{End}(\underline{\mathcal{E}})$ . We will employ the same method for computing both of these and start with  $H_{\mathcal{E},n}$  first.

We re-write the exact sequence (3-1) induced by the helical condition as

$$(6-3) \quad 0 \rightarrow \mathcal{E}_{-n-3} \rightarrow \mathcal{E}_{-n-2}^{\oplus \Delta_{-n-2}} \rightarrow \mathcal{E}_{-n-1}^{\oplus \Delta_{-n-1}} \rightarrow \mathcal{E}_{-n} \rightarrow 0.$$

Bilinearity of the Euler form gives

$$(6-4) \quad \chi(\mathcal{E}_{-n-i}, \mathcal{E}_{-n}) - \Delta_{-n} \chi(\mathcal{E}_{-n-i}, \mathcal{E}_{-n-1}) + \Delta_{-n-2} \chi(\mathcal{E}_{-n-i}, \mathcal{E}_{-n-2}) - \chi(\mathcal{E}_{-n-i}, \mathcal{E}_{-n-3}) = 0$$

which suggests the following result.

**Lemma 6.1.**  $H_{\mathcal{E},n}(t) - \Delta_{-n} t H_{\mathcal{E},n+1}(t) + \Delta_{-n-2} t^2 H_{\mathcal{E},n+2}(t) - t^3 H_{\mathcal{E},n+3}(t) = 1 - t^3$ .

*Proof.* We just compare coefficients of  $t^i$  on both sides. If  $i > 3$ , then all the  $\chi$  terms in (6-4) are given by dimensions of Hom spaces and hence, coefficients of the relative Hilbert series on the left hand side. Thus (6-4) immediately shows the power series on the left is a polynomial of degree  $\leq 3$ . The constant term on the left is also clearly 1 whilst the coefficient of  $t$  is

$$\dim(\mathcal{E}_{-n-1}, \mathcal{E}_{-n}) - \Delta_{-n} \dim(\mathcal{E}_{-n-1}, \mathcal{E}_{-n-1}) = 0.$$

For the coefficients of  $t^2, t^3$ , it is easier to “correct” (6-4) with Ext computations. The co-efficient of  $t^2$  is by Serre duality

$$\Delta_{-n-2} \dim \text{Ext}^1(\mathcal{E}_{-n-2}, \mathcal{E}_{-n-2}) - \dim \text{Ext}^1(\mathcal{E}_{-n-2}, \mathcal{E}_{-n-3}) = 0$$

and that of  $t^3$  is  $-\dim \text{Ext}^1(\mathcal{E}_{-n-3}, \mathcal{E}_{-n-3}) = -1$ .  $\square$

In the “equigenerated” elliptic helices of [10, Section 8] there was a single Hilbert series whereas here, we have three as per the following.

**Proposition 6.2.** *We have  $H_{\mathcal{E},n+3}(t) = H_{\mathcal{E},n}(t)$ .*

*Proof.* We prove 3-periodicity simultaneously for all  $n$  by induction on the degree of the coefficients. The coefficients of negative degree are all zero. For non-negative degrees, Lemma 6.1 expresses any term of  $H_{\mathcal{E},n}(t)$  in terms of the  $\Delta_n$ , which are 3-periodic in  $n$  by Proposition 5.2, and lower order terms in other  $H_{\mathcal{E},m}(t)$  which, by induction are also 3-periodic.  $\square$

**Theorem 6.3.** *Let*

$$(6-5) \quad D(t) := \begin{pmatrix} 1 - t^3 & -\Delta_3 t & \Delta_1 t^2 \\ \Delta_3 t^2 & 1 - t^3 & -\Delta_2 t \\ -\Delta_1 t & \Delta_2 t^2 & 1 - t^3 \end{pmatrix}.$$

Then

$$(6-6) \quad \begin{pmatrix} H_{\mathbb{S},0}(t) \\ H_{\mathbb{S},1}(t) \\ H_{\mathbb{S},2}(t) \end{pmatrix} = \frac{1}{D(t)} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} H_{\mathcal{E},0}(t) \\ H_{\mathcal{E},1}(t) \\ H_{\mathcal{E},2}(t) \end{pmatrix} = \frac{1-t^3}{D(t)} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Furthermore,  $H_{\mathbb{S},n+3}(t) = H_{\mathbb{S},n}(t)$  for all  $n$ .

*Proof.* The Hilbert series for  $\text{End}(\underline{\mathcal{E}})$  is just a re-statement of Lemma 6.1 taking into account the 3-periodicity of Proposition 6.2. To obtain the Hilbert series for  $\mathbb{S}^{nc}(\underline{\mathcal{E}})$  we use the Koszul resolution guaranteed by Bondal-Polshchuk's Theorem 3.7. Abbreviating  $\mathbb{S}$  for  $\mathbb{S}^{nc}(\underline{\mathcal{E}})$ , this is

$$(6-7) \quad 0 \rightarrow e_n \mathbb{S} \rightarrow e_{n+1} \mathbb{S}^{\oplus \Delta_{-n-2}} \rightarrow e_{n+2} \mathbb{S}^{\oplus \Delta_{-n}} \rightarrow e_{n+3} \mathbb{S} \rightarrow \mathbb{S}_{n+3,n+3} \rightarrow 0.$$

Taking alternating sums of dimensions in degree  $n+j$  gives the following analogue of Lemma 6.1:

$$H_{\mathbb{S},n}(t) - \Delta_{-n} t H_{\mathbb{S},n+1}(t) + \Delta_{-n-2} t^2 H_{\mathbb{S},n+2}(t) - t^3 H_{\mathbb{S},n+3}(t) = 1.$$

Now 3-periodicity of  $H_{\mathbb{S},n}(t)$  follows as before as does the formula for the Hilbert series.  $\square$

We do not know if in general, noetherian  $\mathbb{Z}$ -algebras have “sub-exponential” growth as is the case for graded algebras by [20]. However, in our case we do have the following.

**Proposition 6.4.** *Let  $\underline{\mathcal{E}}$  be such that  $\text{End}(\underline{\mathcal{E}})$  is right noetherian. Then  $\underline{\mathcal{E}}$  has polynomial growth and we are in the Markov case.*

*Proof.* By Proposition 5.7, it suffices to assume  $\underline{\mathcal{E}}$  has exponential growth and construct a non-finitely generated submodule of  $e_0 B$  where  $B := \text{End}(\underline{\mathcal{E}})$ . We can use Stephenson-Zhang's argument [20, Theorem 1.2] as follows. From (5-7) in the proof of Proposition 5.7, we see that  $d_n := \dim e_0 B e_{3n}$  grows exponentially so by [20, Lemma 1.1.3], there is a sequence  $l_1 < l_2 < \dots$  such that

$$(6-8) \quad d_{l_s} > \sum_{i < s} d_{l_s - l_i}$$

for all  $s \in \mathbb{N}$ . We show that one can find, inductively on  $s$ , elements  $x_s \in B_{0,3l_s}$  such that  $x_s \notin x_1 B + \dots + x_{s-1} B$ . By the 3-periodicity of Hilbert series in Proposition 6.2, we know that  $\dim_k x_i B e_{3l_s} \leq \dim_k e_{3l_i} B e_{3l_s} = d_{l_s - l_i}$ . Thus our desired  $x_s$  exists by (6-8).  $\square$

As in Artin-Tate-van den Bergh's classic study [3] of regular graded algebras of dimension three, the relationship between the Hilbert series is a manifestation of the fact that we have an embedding of a noncommutative elliptic curve as a cubic in a noncommutative projective plane.

**Corollary 6.5.** *Let  $\underline{\mathcal{E}}$  be an elliptic helix and let  $\pi : \mathbb{S}^{nc}(\underline{\mathcal{E}}) \rightarrow \text{End}(\underline{\mathcal{E}})$  denote the canonical surjection. Then  $\ker \pi$  is generated by a normal family of regular elements of degree three.*

*Proof.* This is a generalisation of [10, Theorem 8.7] to the not necessarily equigenerated case. In fact, the proof there carries over verbatim once we realise that the equigeneration hypothesis was only used to ensure  $H_{\mathcal{E},n}(t) = (1-t^3)H_{\mathbb{S},n}(t)$ . This latter is just Theorem 6.3.  $\square$

7. CONSTRUCTING SEEDS WITH GIVEN  $\Delta_i$ 'S: THE MARKOV CASE

We wish to construct all elliptic helices  $\underline{\mathcal{E}}$  on an elliptic curve  $X$  with polynomial growth. Towards this end, in this section, we construct seeds with given invariants  $\Delta_i$ , in the Markov case.

We will utilize numerous properties of Markov triples, which we briefly review now. We will often use the fact that Markov triples are relatively prime [1, Corollary 3.4]. Another result that will be useful to us is that Markov triples can be *mutated* to form new triples [1, Section 3.1]: if  $(m_1, m_2, m_3)$  is a Markov triple, then so are

$$(7-1) \quad (3m_2m_3 - m_1, m_2, m_3), (m_1, 3m_1m_3 - m_2, m_3), (m_1, m_2, 3m_1m_2 - m_3).$$

**Lemma 7.1.** *If  $m$  is a Markov number (i.e. a component of a Markov triple), then  $m$  isn't divisible by three.*

*Proof.* We know from [8, Lemma 5, p. 28] that any Markov triple can be obtained from the Markov triple  $(1, 1, 1)$  by a sequence of Markov mutations. However, from the formula (7-1), we see that modulo 3, such a mutation only changes the sign of one of the components.  $\square$

We now fix a triple of positive integers  $\Delta_1, \Delta_2, \Delta_3$  which satisfies the ‘‘Markov’’ condition  $\Delta_1^2 + \Delta_2^2 + \Delta_3^2 = \Delta_1\Delta_2\Delta_3$  from (1-2). Note then that  $\Delta_1, \Delta_2, \Delta_3$  are all multiples of 3 and so dividing by 3 we get one of the usual Markov triples. We now ask the converse question, when and how can we construct a *seed*  $[E_i] = (d_i, r_i) \in K_0(X) = \mathbb{Z}^2$  for  $i = 0, 1, 2$  such that  $\chi(E_{i-1}, E_i) = \Delta_i$  for  $i = 1, 2, 3$  where  $E_3$  is the double right mutation of  $E_0$  defined in Section 5. We also want the seed to generate an elliptic helix  $\underline{\mathcal{E}}$  of bundles on  $X$ , so in particular, we insist that the ranks  $r_i$  of the  $E_i$  are positive integers. In this section, we give a complete answer.

If we write  $\vec{d} = \begin{pmatrix} d_0 \\ d_1 \\ d_2 \end{pmatrix}$ , and  $\vec{r} = \begin{pmatrix} r_0 \\ r_1 \\ r_2 \end{pmatrix}$ , then (5-2) gives the following

**Proposition 7.2.** *With the above notation, we have  $\chi(E_{i-1}, E_i) = \Delta_i$  for  $i = 1, 2, 3$  if and only if*

$$(7-2) \quad \vec{d} \times \vec{r} = - \begin{pmatrix} \Delta_2 \\ -\Delta_{02} \\ \Delta_1 \end{pmatrix}$$

where  $\Delta_{02} = \Delta_1\Delta_2 - \Delta_3$ .

Note that  $\vec{\Delta} = (\Delta_2, \Delta_3, \Delta_1) \mapsto (\Delta_2, \Delta_1\Delta_2 - \Delta_3, \Delta_1)$  is a type of Markov mutation at the middle coordinate by (7-1).

Now, let  $A$  be the matrix defined in Proposition 5.3 so the ranks of the helix components  $E_i$  are given by  $A^n \vec{r}, n \in \mathbb{Z}$ .

**Proposition 7.3.** *Suppose that the seed given by  $\vec{d}, \vec{r}$  satisfies Equation (7-2) and  $A^n \vec{r}$  consists of positive integers for all  $n$ . Then*

$$(7-3) \quad \vec{\Delta} := \begin{pmatrix} \Delta_2 \\ \Delta_3 \\ \Delta_1 \end{pmatrix} = 3\vec{r} \quad \text{or} \quad \vec{r}$$

In the former case,  $\vec{r}$  is a Markov triple, that is,

$$(7-4) \quad r_0^2 + r_1^2 + r_2^2 = 3r_0r_1r_2,$$

Furthermore, we have

$$(7-5) \quad A^n \vec{d} = \vec{d} + 3n\vec{\Delta}.$$

*Proof.* From Proposition 5.4 we know  $A$  has eigenvalue 1 with multiplicity 3. Furthermore, by Lemma 5.5(2),  $\ker(A - I) = \mathbb{Q}\vec{\Delta}$  so  $A$  must consist of a single Jordan block. Thus  $\ker(A - I)^2 = \text{im}(A - I)$  is 2-dimensional. We now claim  $\ker(A - I)^2 = \vec{\nabla}^\perp$  where  $\vec{\nabla} = \begin{pmatrix} \Delta_2 \\ \Delta_3 - \Delta_1 \Delta_2 \\ \Delta_1 \end{pmatrix}$ . To prove the claim, we observe that

since  $v \in \ker(A - I)^2 = \text{im}(A - I)$  implies  $v = (A - I)w$ ,  $\vec{\nabla} \cdot v = 0$  by Lemma 5.5(1).

Thus for Equation (7-2) to hold, we must have  $\vec{r} \in \ker(A - I)^2$ , and hence in  $\text{im}(A - I)$  so  $A\vec{r} = \vec{r} + \alpha\vec{\Delta}$  for some  $\alpha \in \mathbb{Q}$ . But then  $A^n \vec{r} = \vec{r} + n\alpha\vec{\Delta}$  which must have negative entries for some  $n$  unless  $\alpha = 0$ . We have thus shown that  $\vec{r} \in \ker(A - I) = \mathbb{Q}\vec{\Delta}$ . Now the greatest common divisor of all the coordinates of  $\vec{\nabla}$  is 3, so either  $\vec{r} = \vec{\Delta}$  or  $\frac{1}{3}\vec{\Delta}$ .

Prompted by the first column of (5-4), we let  $\vec{\delta} := \frac{3}{\Delta_2} \begin{pmatrix} 0 \\ \Delta_1 \\ \Delta_1 \Delta_2 - \Delta_3 \end{pmatrix}$ . A direct computation shows that  $\vec{\delta} \times \vec{r} = -\vec{\nabla}$  so  $\vec{d} = \vec{\delta} + \beta\vec{\Delta}$  for some  $\beta \in \mathbb{Q}$ . It follows by direct computation that

$$(A - I)\vec{d} = (A - I)\vec{\delta} = 3\vec{\Delta}$$

giving (7-5).  $\square$

*Remark 7.4.* Recall from Proposition 5.3 that if Equation (7-3) holds, then  $A\vec{r} = \vec{r}$  so positivity of ranks is guaranteed.

Consider the Markov triple  $\vec{\rho} = (\rho_0, \rho_1, \rho_2)^T = \frac{1}{3}\vec{\Delta}$ . We define three ‘‘signed Markov’’ mutations as follows. For any vector  $\vec{v} = (v_0, v_1, v_2)^T \in \mathbb{Q}^3$ , let

$$\mu_0(\vec{v}) = \begin{pmatrix} v_0 + 3v_1v_2 \\ v_1 \\ v_2 \end{pmatrix}, \quad \mu_1(\vec{v}) = \begin{pmatrix} v_0 \\ v_1 - 3v_0v_2 \\ v_2 \end{pmatrix}, \quad \mu_2(\vec{v}) = \begin{pmatrix} v_0 \\ v_1 \\ v_2 + 3v_0v_1 \end{pmatrix},$$

Recall that the absolute values of the coordinates of  $\mu_1(\vec{\rho})$  are Markov triple since  $\vec{\rho}$  is. The choice of signs means that the Markov equation (7-4) gives the conditions

$$(7-6) \quad \vec{\rho} \perp \mu_1(\vec{\rho}) =: \vec{\mu}_1, \quad \mu_1(\vec{\rho}) \perp \mu_0(\mu_1(\vec{\rho})) =: \vec{\mu}_0, \quad \mu_1(\vec{\rho}) \perp \mu_2(\mu_1(\vec{\rho})) =: \vec{\mu}_2$$

We now consider the two cases in Proposition 7.3 separately:

- (1) Case 1:  $\vec{r} = \vec{\rho}$ . We solve Equation (7-2) for  $\vec{d}$  which now becomes

$$(7-7) \quad \vec{d} \times \vec{\rho} = -3\vec{\mu}_1$$

Any solution  $\vec{d}$ , if it exists, is unique up to an additive multiple of  $\vec{r}$ . Also,  $\vec{d}$  must be orthogonal to  $\vec{\mu}_1$  so natural choices are linear combinations of  $\vec{\mu}_0, \vec{\mu}_2$ . One computes

$$(7-8) \quad \vec{\mu}_0 \times \vec{\rho} = -3r_2^2\vec{\mu}_1, \quad \vec{\mu}_2 \times \vec{\rho} = -3r_0^2\mu_1(\vec{\rho})$$

- (2) Case 2:  $\vec{r} = 3\vec{\rho}$ . A similar analysis as in Case 1 yields

$$(7-9) \quad \vec{d} \times \vec{\rho} = -\vec{\mu}_1$$

and

$$(7-10) \quad \vec{\mu}_0 \times \vec{\rho} = -3\rho_2^2\vec{\mu}_1, \quad \vec{\mu}_2 \times \vec{\rho} = -3\rho_0^2\mu_1(\vec{\rho})$$

Recall that the coordinates of a Markov triple are relatively prime. Thus we have the following

**Theorem 7.5.** *Let  $\vec{r} \in \mathbb{N}^3$  be a Markov triple and set  $\vec{\Delta} := (\Delta_2, \Delta_3, \Delta_1) = 3(r_0, r_1, r_2)$ . Then one can find degrees  $\vec{d} \in \mathbb{Z}^3$  such that the seed in  $K_0(X)$  defined by  $\vec{d}, \vec{r}$  has determinants given by  $\vec{\Delta}$ . In fact, given any integers  $a, b$  such that  $ar_0^2 + br_2^2 = 1$  we have the solution  $\vec{d} = a\vec{\mu}_2 + b\vec{\mu}_0$  and all other solutions can be obtained from this by adding appropriate multiples of  $\vec{r}$ .*

*Proof.* It remains only to check  $\gcd(d_i, r_i) = 1$ . But any non-trivial common divisor of  $d_i, r_i$  would also divide two of the components of  $\mu_1(\vec{\Delta})$ . This contradicts the fact that the absolute values of the components of  $\mu_1(\vec{\Delta})$  are again a Markov triple.  $\square$

It remains only to eliminate the possibility  $\vec{r} = 3\vec{\rho}$ .

**Theorem 7.6.** *Let  $(E_0, E_1, E_2)$  be a seed that generates an elliptic helix  $\underline{\mathcal{E}}$  on  $X$  with polynomial growth. Then*

$$(\text{rk } E_0, \text{rk } E_1, \text{rk } E_2) =: \vec{r}$$

*is a Markov triple and  $\vec{\Delta} = 3\vec{r}$ .*

*Proof.* We know from Proposition 5.7 that  $\vec{\Delta} = 3\vec{\rho}$  for some Markov triple  $\vec{\rho}$ . Furthermore, Proposition 7.3 reduces the possibilities to  $\vec{r} = \vec{\rho}$  or  $3\vec{\rho}$ . We seek a contradiction to the latter case by and so assume  $\vec{r} = 3\vec{\rho}$ . We solve now (7-9). Since  $\vec{\mu}_0, \vec{\mu}_2$  are linearly independent, as one can check by explicit computation, we can find integers  $b_0, b_2, m$  such that  $\vec{d} = \frac{b_0}{m}\vec{\mu}_0 + \frac{b_2}{m}\vec{\mu}_2$ . We may of course, assume that  $m$  is as small a positive integer as possible. From the computations in Equation (7-10), this reduces to

$$(7-11) \quad m = 3(b_0\rho_2^2 + b_2\rho_0^2)$$

so in particular,  $m$  is divisible by 3. Writing  $\rho_{02}$  for  $\rho_1 - 3\rho_0\rho_2$  we have

$$(7-12) \quad \vec{\mu}_1 = \begin{pmatrix} \rho_0 \\ \rho_{02} \\ \rho_2 \end{pmatrix}, \quad \vec{\mu}_0 = \begin{pmatrix} \rho_0 + 3\rho_{02}\rho_2 \\ \rho_{02} \\ \rho_2 \end{pmatrix}, \quad \vec{\mu}_2 = \begin{pmatrix} \rho_0 \\ \rho_{02} \\ \rho_2 + 3\rho_{02}\rho_0 \end{pmatrix}$$

We now use integrality of the coordinates of  $\vec{d}$ . From the middle coordinate, we know that  $(b_0 + b_2)\rho_{02} \in m\mathbb{Z}$ . Now  $-\rho_{02}$  is a Markov number so cannot be divisible by 3 by Lemma 7.1. Thus  $3|b_0 + b_2$ .

Now

$$d_0 = (b_0 + b_2)\rho_0 + 3b_0\rho_2\rho_{02} \equiv 0 \pmod{3}.$$

This contradicts stability of  $\mathcal{E}_0$  since  $3|r_0$  too.  $\square$

## 8. ELLIPTIC HELICES OF MARKOV TYPE ARE AMPLE

The goal of this section is to show that three-periodic elliptic helices on  $X$  with polynomial growth are ample. As a consequence, we prove, in Corollary 8.2, that endomorphism algebras and their quadratic covers are noetherian. In what follows, we refer to such helices as elliptic helices of Markov type.

**Proposition 8.1.** *Suppose  $\underline{\mathcal{E}} = (\mathcal{E}_i)_{i \in \mathbb{Z}}$  is an elliptic helix of Markov type. Let  $d_i = \deg \mathcal{E}_i$ , let  $r_i = \text{rank } \mathcal{E}_i$ , and suppose  $\mu_i := d_i/r_i$ . Then  $\lim_{n \rightarrow -\infty} \mu_n = -\infty$ , and  $\underline{\mathcal{E}}$  is ample for  $\text{Coh } X$ .*

*Proof.* We continue the notation from Section 7. From Proposition 7.3 we know that  $\vec{r}$  is also a 1-eigenvector for  $A$  so  $r_n$  is 3-periodic in  $n$  and in particular bounded. Also Proposition 7.3 (7-5) shows that  $d_n \rightarrow -\infty$  as  $n \rightarrow -\infty$  so  $\lim_{n \rightarrow -\infty} \mu_n = -\infty$ . We now use the ampleness criterion of Lemma 4.1 and so consider  $\mathcal{M} \in \text{Coh} X$  and show  $\text{Ext}_X^1(\mathcal{E}_i, \mathcal{M}) = 0$  for  $i \ll 0$ . Filtering  $\mathcal{M}$ , we may assume by induction that  $\mathcal{M}$  is a simple sheaf with slope  $\mu$ . Then by Serre duality we have

$$\text{Ext}_X^1(\mathcal{E}_i, \mathcal{M}) \simeq \text{Hom}_X(\mathcal{M}, \mathcal{E}_i)^* = 0$$

as soon as the slope of  $\mathcal{E}_i$  drops below  $\mu$ .  $\square$

**Corollary 8.2.** *If  $\underline{\mathcal{E}}$  is an elliptic helix of Markov type, then  $\text{End}(\underline{\mathcal{E}})$  and  $\mathbb{S}^{nc}(\underline{\mathcal{E}})$  are noetherian.*

*Proof.* For the first assertion, it remains to check the left noetherian condition, and this will follow from Proposition 2.3 provided we can prove the dual helix is ample. By Proposition 8.1, it thus suffices to show the dual helix is an elliptic helix of Markov type. Since every triple of components of the dual helix has ranks a Markov triple, this is clear in light of the proof of [10, Theorem 7.23].

The fact that  $\mathbb{S}^{nc}(\underline{\mathcal{E}})$  is right noetherian follows immediately from Corollary 6.5 and Proposition 2.4. To prove it is left noetherian, it suffices to prove that the opposite algebra is right noetherian. However, we know the opposite algebra of  $\text{End}(\underline{\mathcal{E}})$  is right noetherian by the first part of the proof. Thus, the result holds by considering the short exact sequence from Corollary 6.5 applied to opposite algebras and again invoking Proposition 2.4.  $\square$

Let  $\underline{\mathcal{E}}$  be an elliptic helix of Markov type and  $d_i, r_i$  denote the degree and rank of  $\mathcal{E}_i$  as usual. Here we give a more direct computation of the Hilbert series of  $\text{End}(\underline{\mathcal{E}})$ . Let  $i', j' \in \{0, 1, 2\}$  and  $m \in \mathbb{N}$ . From Proposition 7.3 (7-5) we have  $d_{i'+3m} = d_{i'} + 3m\Delta_{i'-1}$ . Now (5-7) and the fact that  $3r_j = \Delta_{j-1}$  shows for  $j' < i' + 3m$  that

$$\dim_k \text{Hom}(\mathcal{E}_{j'}, \mathcal{E}_{i'+3m}) = \begin{vmatrix} d_{i'} + 3m\Delta_{i'-1} & d_{j'} \\ r_{i'} & r_{j'} \end{vmatrix} = \chi(\mathcal{E}_{j'}, \mathcal{E}_{i'}) + m\Delta_{i'-1}\Delta_{j'-1}$$

## 9. MUTATING ELLIPTIC HELICES

Given an elliptic helix  $\underline{\mathcal{E}}$  on an elliptic curve  $X$ , we construct a new elliptic helix by mutating every third bundle. We show that in the Markov case, the triple of ranks of bundles changes correspondingly by a Markov mutation.

**Proposition 9.1.** *Let  $\underline{\mathcal{E}}$  be an elliptic helix of bundles on  $X$ .*

- (1) *Right (or left) mutating every third bundle yields another elliptic helix  $\underline{\mathcal{E}'}$  so in particular, the following is an elliptic helix*

$$(9-1) \quad \dots, \mathcal{E}_1, R_{\mathcal{E}_1}\mathcal{E}_0, \mathcal{E}_2, \mathcal{E}_4, R_{\mathcal{E}_4}\mathcal{E}_3, \mathcal{E}_5, \dots$$

- (2) *Suppose we are in the Markov case of Theorem 7.6 so in particular, the rank triple of the seed  $\vec{r} := (r_0 = \text{rk } \mathcal{E}_0, r_1 = \text{rk } \mathcal{E}_1, r_2 = \text{rk } \mathcal{E}_2)$  is a Markov triple. Then the seed of (9-1) has rank triple  $(r_1, 3r_2r_1 - r_0, r_2)$*

*Proof.* Note first that shifting the indices of an elliptic helix by an additive constant yields another elliptic helix, so part (1) will be proved if we can show that the collection of stable bundles in (9-1) is an elliptic helix. First note that stable bundles  $\mathcal{E}$  on  $X$  are spherical objects in the sense of [11, Definition 8.1] so the

right and left mutations  $R_{\mathcal{E}}, L_{\mathcal{E}}$  extend to inverse auto-equivalences of  $D^b(X)$ , by [11, Proposition 8.6]. Denoting these extensions also by  $R_{\mathcal{E}}, L_{\mathcal{E}}$ , [11, Lemma 8.21] readily gives the following useful formulae for stable bundles  $\mathcal{E}, \mathcal{F}$ :

$$(9-2) \quad R_{\mathcal{F}} \circ R_{L_{\mathcal{F}}\mathcal{E}} = R_{\mathcal{E}} \circ R_{\mathcal{F}}, \quad R_{\mathcal{F}} \circ R_{\mathcal{E}} = R_{R_{\mathcal{F}}\mathcal{E}} \circ R_{\mathcal{F}}.$$

We first show that doubly right mutating the  $i$ -th bundle  $\mathcal{E}'_i$  in (9-1) gives the  $i+3$ -rd bundle  $\mathcal{E}'_{i+3}$ . By symmetry, it suffices to do this for  $i = 0, 1, 2$ . Since  $\mathcal{E}$  is an elliptic helix we have

$$R_{\mathcal{E}_4}R_{\mathcal{E}_2}(R_{\mathcal{E}_1}\mathcal{E}_0) = R_{\mathcal{E}_4}\mathcal{E}_3$$

as desired. Using (9-2) we find

$$R_{R_{\mathcal{E}_4}\mathcal{E}_3}R_{\mathcal{E}_4}\mathcal{E}_2 = R_{\mathcal{E}_4}R_{\mathcal{E}_3}\mathcal{E}_2 = \mathcal{E}_5.$$

Finally, note that since  $\underline{\mathcal{E}}$  is an elliptic helix,  $R_{\mathcal{E}_1}\mathcal{E}_0 = L_{\mathcal{E}_2}\mathcal{E}_3$  hence

$$R_{\mathcal{E}_2}R_{L_{\mathcal{E}_2}\mathcal{E}_3}\mathcal{E}_1 = R_{\mathcal{E}_3}R_{\mathcal{E}_2}\mathcal{E}_1 = \mathcal{E}_4$$

proving part (1).

For part (2), just note that

$$\mathrm{rk} R_{\mathcal{E}_1}\mathcal{E}_0 = \Delta_1 r_1 - r_0 = 3r_2 r_1 - r_0.$$

□

*Definition 9.2.* We say that two elliptic helices  $\underline{\mathcal{E}}, \underline{\mathcal{E}'}$  are *mutations* of each other if they can be gotten from each other by a series of alterations of the form given in (1) of Proposition 9.2.

**Corollary 9.3.** *Any elliptic helix  $\underline{\mathcal{E}}$  which is of Markov type, that is, as described in Theorem 7.6, is a mutation of an elliptic helix consisting of line bundles. Thus,  $\mathrm{ProjS}^{nc}(\underline{\mathcal{E}})$  is equivalent to an elliptic quantum plane.*

*Proof.* Note that there are three possible ways you can mutate every third bundle in  $\mathcal{E}$ , and by Proposition 9.1 (2), they correspond to the three possible mutations of the Markov triple of ranks. Since we can always find a sequence of these mutations of Markov triples to reduce to the triple  $(1, 1, 1)$  by [8, Lemma 5, p. 28] we are done with a proof of the first statement.

For the second statement, it suffices to show that noncommutative symmetric algebras of mutated elliptic helices of Markov type (i.e. quadratic covers of the associated endomorphism algebras) are Proj-equivalent. To this end, first note that if  $A$  is a  $\mathbb{Z}$ -algebra and  $i \in \mathbb{Z}$ , then the shifted  $\mathbb{Z}$ -algebra  $A(i)$  defined by  $A(i)_{jk} = A_{j+i, k+i}$  has an equivalent graded module category (the equivalence is shifting of modules by  $i$ ). Suppose  $\underline{\mathcal{E}}$  is an elliptic helix of Markov type and  $\underline{\mathcal{E}'}$  is the elliptic helix (9-1). Let  $S$  and  $S'$  denote the corresponding noncommutative symmetric algebras. Then, for appropriate  $i$ ,  $S(i)$  and  $S'(i)$  have equal 3-Veroneses. Since both are right noetherian by Corollary 8.2, they are Proj-equivalent by [18, Lemma 3.5]. □

## 10. CONSTRUCTING ELLIPTIC HELICES $\underline{\mathcal{E}}$ WITH NON-NOETHERIAN $\mathrm{End}(\underline{\mathcal{E}})$

In [10, Theorem 8.1], we constructed a single family of elliptic helices  $\underline{\mathcal{E}}$  whose endomorphism ring was non-noetherian. It happens to be equigenerated which made the analysis amenable to the simpler methods used there. In this section, we wish to construct more examples with  $\mathrm{End}(\underline{\mathcal{E}})$  non-noetherian including some which are not equigenerated.

The key question is, when does a seed  $(E_0, E_1, E_2)$  generate an elliptic helix. The question is purely numerical, so we start with the corresponding numerical seed  $([E_0], [E_1], [E_2]) = ((\frac{d_0}{r_0}), (\frac{d_1}{r_1}), (\frac{d_2}{r_2}))$ .

Recall Equation (5-3) determines all the  $d_i, r_i$  in terms of the numerical seed and the  $\Delta_i$ . The numerical seed generates an elliptic helix if all the  $r_i, r'_i > 0$  by [10, Theorem 7.14]. Our approach to a necessary criterion to guarantee this is that Equation (5-6) ensures that  $r_{i+3n}$  is either a monotonic function of  $n$  or is concave up. We will look for seeds which give a concave up function  $r_n$  with a minimum at  $n = 1$ .

**Proposition 10.1.** *Suppose a numerical seed with  $r_0, r_2 \geq r_1$  generates a sequence  $r_i$  which is increasing for  $i \geq 1$ , decreasing for  $i < 1$  and such that  $r'_1, r'_2 > 0$ . Then the seed generates an elliptic helix.*

*Proof.* Note first all  $r_i \geq 0$ . Now  $r_i$  is increasing for  $i \geq 1$  so

$$r'_{i+2} = \Delta_{i+1}r_{i+1} - r_i > 0$$

and a similar argument shows that  $r'_j > 0$  for all  $j \leq 0$ .  $\square$

We wish to reduce checking piecewise monotonicity of  $r_i$  to some finite condition as follows.

**Proposition 10.2.** *Suppose the numerical seed has  $r_0, r_2 \geq r_1$  and  $r'_1, r'_2 > 0$ . Suppose there exists a real number  $l > 1$  such that*

$$(10-1) \quad \Delta_i - \frac{\Delta_j}{l} > l, \quad \text{for all distinct } i, j.$$

*If  $\frac{r_0}{r_1}, \frac{r_2}{r_1} > l$  then the seed generates an elliptic helix.*

*Proof.* We first show by induction that  $\frac{r_{i+1}}{r_i} > l$  for  $i \geq 1$ . Note Equation (5-3) gives for  $i \geq 0$

$$\frac{r_{i+3}}{r_{i+2}} = \frac{r_i}{r_{i+2}} - \Delta_{i+1} \frac{r_{i+1}}{r_{i+2}} + \Delta_i > \Delta_i - \frac{\Delta_{i+1}}{l} > l.$$

A similar argument shows  $r_i$  is descending for  $i < 1$  so Proposition 10.1 applies.  $\square$

The approach suggests a natural starting point to look for seeds generating elliptic helices is to try  $r_1 = 1$  in which case  $L_1$  is a line bundle which we may as well normalise to be  $\mathcal{O}$ .

**Example 10.3.** Here we seek equigenerated examples, that is, where  $\Delta_i = \Delta$  for all  $i$ . We thus start with the numerical seed  $((\frac{-\Delta}{r_0}), (\frac{0}{1}), (\frac{\Delta}{r_2}))$  so  $\Delta_1 = \Delta_2 = \Delta$ . From (5-2), we also want

$$\Delta(r_0 + r_2) = \Delta_{02} = \Delta^2 - \Delta \Leftrightarrow \Delta = r_0 + r_2 + 1.$$

We apply Proposition 10.2 with  $l = 2 - \varepsilon$  where  $\varepsilon > 0$  is sufficiently small. Then (10-1) becomes

$$\Delta > \frac{2 - \varepsilon}{1 - (2 - \varepsilon)^{-1}} = \frac{(2 - \varepsilon)^2}{1 - \varepsilon} \rightarrow 4$$

as  $\varepsilon \rightarrow 0$ . Note that  $r'_2 = \Delta - r_0 = r_2 + 1 > 0, r'_1 = \Delta - r_2 = r_0 + 1 > 0$ . Thus the numerical seed  $((\frac{-r_0 - r_2 - 1}{r_0}), (\frac{0}{1}), (\frac{r_0 + r_2 + 1}{r_2}))$  generates an elliptic helix whenever  $r_0, r_2 \geq 2$  and  $\gcd(r_0 + r_2 + 1, r_i) = 1$  for  $i = 0, 2$ . Note that these examples are distinct from those constructed in [10].

We can generalise this to non-equigenerated examples as follows. Consider the numerical seed  $((\begin{smallmatrix} -r_0-r_2-a \\ r_0 \end{smallmatrix}), (\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}), (\begin{smallmatrix} r_0+r_2+a \\ r_2 \end{smallmatrix}))$  for some  $a \in \mathbb{Z}_{\geq 1}$  so  $\Delta_1 = \Delta_2 = r_0 + r_2 + a$  but now  $\Delta_0 = a(r_0 + r_2 + a)$ . Picking  $l = a + 1 - \varepsilon$  in Proposition 10.1, the above analysis shows we obtain an elliptic helix whenever  $r_0, r_2 \geq a + 1$ ,  $r_0 + r_2 > a^2 + a + 1$  and, of course  $\gcd(r_0 + r_2 + a, r_i) = 1$  for  $i = 0, 2$ .

**Example 10.4.** Consider the numerical seed  $((\begin{smallmatrix} -d-a-1 \\ d+a-r \end{smallmatrix}), (\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}), (\begin{smallmatrix} d \\ r \end{smallmatrix}))$  where  $a, d, r \in \mathbb{N}$  are chosen so that  $\gcd(d + a + 1, d + a - r) = 1 = \gcd(d, r)$ . Clearly we must have  $d + a - r > 0$ . Note  $\Delta_0 = d - (a + 1)r$ ,  $\Delta_1 = d + a + 1$ ,  $\Delta_2 = d$ . To apply Proposition 10.2 we first need

$$r'_1 = \Delta_2 r_1 - r_2 = d - r > 0, \quad r'_2 = \Delta_1 r_1 - r_0 = r + 1 > 0.$$

We also need to impose the condition (10-1) for some  $l$ . Picking  $l = 2$ , the set of inequalities amount to the strictest one

$$\min_i \Delta_i - \max_j \frac{\Delta_j}{2} = d - (a + 1)r - \frac{d + a + 1}{2} > 2$$

which is equivalent to the condition  $d > (a + 1)(2r + 1) + 4$ . Finally, we impose the condition  $\min\{d + a - r, r\} > 2$ . When these hold, we obtain an elliptic helix.

These examples suggests that elliptic helices are relatively plentiful, but finding them is not so easy, let alone giving a complete classification.

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